Interior Point Methods

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Outline

- Interior point method: basic idea
- Log barrier function
- Central path
- Barrier (interior point) method
- A hierarchy of convex optimization
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- Interior point method: basic idea
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Inequality constrained minimization

- Convex program with inequality constraints

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad A x = b,
\end{align*}
\]

- \( f_0, \ldots, f_m : \mathbb{R}^n \rightarrow \mathbb{R} \) convex, twice differentiable
- \( A \in \mathbb{R}^{p \times n} \) with rank \( p < n \)
- An optimal \( x^* \) exists
Basic idea: how to solve?

- Recall 1: equality constrained QP

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2}x^T P x + q^T x + r \\
\text{subject to} & \quad Ax = b,
\end{align*}
\]

- KKT conditions

\[
Ax^* = b, \quad Px^* + q + A^T \nu^* = 0,
\]

- Directly solve

\[
\begin{bmatrix}
P & A^T \\
A & 0
\end{bmatrix}
\begin{bmatrix}
x^* \\
\nu^*
\end{bmatrix} =
\begin{bmatrix}
-q \\
b
\end{bmatrix}.
\]
Basic idea: how to solve?

- Recall 2: Newton’s method with equality constraints
  - A quadratic approximation at current $x$
    
    
    $$\begin{align*}
    \text{minimize} & \quad \hat{f}(x + v) = f(x) + \nabla f(x)^T v + (1/2)v^T \nabla^2 f(x)v \\
    \text{subject to} & \quad A(x + v) = b
    \end{align*}$$

  - Solve $v$ (i.e., $\Delta x$)
    
    $$\begin{bmatrix}
    \nabla^2 f(x) & A^T \\
    A & 0
    \end{bmatrix}
    \begin{bmatrix}
    v \\
    w
    \end{bmatrix}
    =
    \begin{bmatrix}
    -\nabla f(x) \\
    0
    \end{bmatrix}$$

  - Need line search, since $\hat{f}$ is an approximation

  - What if starting with an infeasible $x$?
Basic idea: how to solve?

- Convex program with inequality constraints

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b,
\end{align*}
\]

- Decompose into a sequence of equality constrained problem
- Each solved by Newton’s method with equality constraints
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Logarithmic barrier function

- Reformulate inequality constraints

\[
\begin{align*}
\text{minimize} & \quad f_0(x) + \sum_{i=1}^{m} I_-(f_i(x)) \\
\text{subject to} & \quad A x = b
\end{align*}
\]

- where

\[
I_-(u) = \begin{cases} 
0 & u \leq 0 \\
\infty & u > 0.
\end{cases}
\]

- But indicator function is difficult to optimize
Logarithmic barrier function

- An alternative: $-(1/t) \log(-u)$
  - A smooth approximation to $I_-$
  - Improves as $t \to \infty$
Logarithmic barrier function

- Approximation by log barrier function

\[
\begin{aligned}
\text{minimize} & \quad f_0(x) + \sum_{i=1}^{m} -(1/l) \log(-f_i(x)) \\
\text{subject to} & \quad Ax = b.
\end{aligned}
\]

- Log barrier function \( \phi(x) = -\sum_{i=1}^{m} \log(-f_i(x)) \)
  - Convex
  - Twice differentiable

\[
\nabla \phi(x) = \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla f_i(x)
\]

\[
\nabla^2 \phi(x) = \sum_{i=1}^{m} \frac{1}{f_i(x)^2} \nabla f_i(x)\nabla f_i(x)^T + \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla^2 f_i(x)
\]
Logarithmic barrier function

- Approximation by log barrier function

\[
\begin{align*}
\text{minimize} & \quad f_0(x) + \sum_{i=1}^{m} -(1/t) \log(-f_i(x)) \\
\text{subject to} & \quad Ax = b.
\end{align*}
\]

- Log barrier function \( \phi(x) = - \sum_{i=1}^{m} \log(-f_i(x)) \)

\[
\begin{align*}
\text{minimize} & \quad f_0 + \frac{1}{t} \phi \\
\end{align*}
\]

- Rewrite (for a fixed \( t > 0 \))

\[
\begin{align*}
\text{minimize} & \quad tf_0(x) + \phi(x) \\
\text{subject to} & \quad Ax = b
\end{align*}
\]
Logarithmic barrier function

- Solve a sequence of problems

\[
\begin{align*}
\text{minimize} & \quad tf_0(x) + \phi(x) \\
\text{subject to} & \quad Ax = b
\end{align*}
\]

- Increase $t$ step by step
- Why not a large $t$ at the beginning?
  - Difficult to solve via Newton’s method
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Central Path

- For any \( t > 0 \), define \( x^*(t) \) as the solution to:

\[
\begin{align*}
\text{minimize} & \quad tf_0(x) + \phi(x) \\
\text{subject to} & \quad Ax = b
\end{align*}
\]

- Central path is the set:

\[
\{ x^*(t) \mid t > 0 \}
\]
Central Path for LP

- Example: an LP with inequality

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad a_i^T x \leq b_i, \quad i = 1, \ldots, 6
\end{align*}
\]
Central Path: Dual points

- For any $t > 0$, define $x^*(t)$ as the solution to:

  $$\begin{align*}
  \text{minimize} & \quad tf_0(x) + \phi(x) \\
  \text{subject to} & \quad Ax = b
  \end{align*}$$

- $x^*(t)$ is feasible to original problem
  - Not optimal
  - Leads to a dual feasible point $v^*(t)$
  - But duality gap for $x^*(t)$ is bounded: $m/t$ !
  - Useful as stop criteria 😊
KKT interpretations of Central Path

- Consider the problem for a fixed $t$:
  
  \[
  \begin{align*}
  &\text{minimize} & & tf_0(x) + \phi(x) \\
  &\text{subject to} & & Ax = b
  \end{align*}
  \]

- KKT conditions (see textbook)
  
  \[
  \begin{align*}
  Ax &= b, \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
  \lambda &\geq 0 \\
  \nabla f_0(x) + \sum_{i=1}^m \lambda_i \nabla f_i(x) + A^T \nu &= 0 \\
  -\lambda_i f_i(x) &= 1/t, \quad i = 1, \ldots, m.
  \end{align*}
  \]

- Only one difference to KKT for original problem
  
  - Converge to original problem as $t \to \infty$
Force field interpretations

- Consider for $t > 0$ (ignoring $Ax=b$)

\[
\text{minimize} \quad tf_0(x) - \sum_{i=1}^{m} \log(-f_i(x))
\]

- $tf_0(x)$: potential with the force $F_0(x) = -t \nabla f_0(x)$

- $- \log(-f_i(x))$: potential with the force $F_i(x) = (1/f_i(x)) \nabla f_i(x)$

- Force balance at optimal $x^*(t)$

\[
F_0(x^*(t)) + \sum_{i=1}^{m} F_i(x^*(t)) = 0
\]
Force field for LP

- Consider the LP
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Barrier Method

\textbf{given} strictly feasible $x, t := t^{(0)} > 0, \mu > 1$, tolerance $\epsilon > 0$.

\textbf{repeat}

1. \textit{Centering step}. Compute $x^*(t)$ by minimizing $tf_0 + \phi$, subject to $Ax = b$.
2. \textit{Update}. $x := x^*(t)$.
3. \textit{Stopping criterion}. \textbf{quit} if $m/t < \epsilon$.
4. \textit{Increase t}. $t := \mu t$.

- Solve for a sequence of $t$
- Update (inner loops): Newton’s, warm starting
- $\mu$ : large $\rightarrow$ fewer outer loops, more inner loops
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A hierarchy of convex optimization

- QP with equality constraints
  - Directly solve via KKT conditions
- Convex prog. with equality constraints
  - Form a quadratic approximation at $x$
  - Solve the approximation (Newton’s with equality)
  - Line search along the Newton step
- Interior point method
  - Form a sequence of equality constrained problems
  - Newton’s method with equality constraints
Thanks