ON COMBINATORIAL PROPERTIES OF MATRICES

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The starting point of the present work is the following theorem due to Dénes König:

> If the elements of a matrix are partly zeros and partly numbers different from zero (or independent variables), then the minimum number of lines[1] that contain all of the non-zero elements of the matrix is equal to the maximum number of non-zero elements that can be chosen with no two on the same line.

König, in proving this theorem by graph-theoretical means, arrived at the theorem in a graph-theoretical formulation which has connections with other questions of graph theory.[2]

A special case of this theorem is equivalent to the following result of Frobenius on determinants.[3] For the determinant of an n by n matrix with elements that are partly zero and partly independent variables to vanish identically,[4] it is necessary and sufficient that at least $n + 1$ lines have all zeros in common. This condition is sufficient since the realization of this situation for an n by n matrix with $2n$ lines means that all of the non-vanishing elements are contained in the remaining $n - 1$ lines and thus, according to the theorem above, no more than $n - 1$ different non-vanishing elements can be chosen with no two on the same line.

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[1] The name line applies to either the rows or columns of the matrix.

[2] König presented this theorem to the Társulat (society) in March, 1931, and it will appear in his forth-coming book on the theory of graphs. (Theorie der Graphen, New York: Chelsea Publ. Co., 1950)

[3] G. Frobenius: Über zerlegbare Determinanten, Sitz. d. Berl. Ak. 1917, I. pp. 274-77.

[4] Such a determinant is said to vanish identically if all of the terms of the expansion of the determinant are identically zero.

That is to say, all of the terms of the expansion of the determinant are zero. If the condition is not satisfied, that is, if no more than n lines have all zeros in common, then the non-vanishing elements obviously occupy at least n lines and thus, according to the theorem above, it is possible to choose n distinct non-vanishing elements, no two of which lie on the same line. Hence the determinant does not vanish identically. Frobenius gave a proof of this special case of the theorem by algebraic methods.

The aforementioned theorem is given a new proof in §1 and is generalized in §2 to a theorem in the following form:

I. If (a_{ij}) is a given n by n matrix with non-negative integers as elements, and if λ_i and μ_j are non-negative integers such that

(1) $$\lambda_i + \mu_j \geq a_{ij} \qquad (i, j = 1, 2, \ldots, n),$$

then

(2) $$\min \sum_{k=1}^{n} (\lambda_k + \mu_k) = \max(a_{1v_1} + a_{2v_2} + \ldots + a_{nv_n}),$$

where $v_1, v_2, \ldots, v_n$ runs through all permutations of $1, 2, \ldots, n$.

Taking the special case, in which the given elements a_{ij} only assumes the values 0 and 1, the theorem of Konig mentioned above follows as an obvious consequence.

Further, a theorem is established in §3 which stands in a dual relation to the preceding theorem. Namely, if (δ_{ij}^{ρ}), with $\rho = 1, 2, \ldots, n!$, denotes all distinct n by n matrices that arise from the unit matrix through permutations of the rows (or columns), then the following theorem is valid:

II. If (a_{ij}) is a given n by n matrix with non-negative integers as elements, and if v_ρ are non-negative integers such that

(3) $$\sum_{\rho=1}^{n!} v_\rho \delta_{ij}^{\rho} \geq a_{ij} \qquad (i, j = 1, 2, \ldots, n),$$

then

$$(4)\ \min \sum_{\rho=1}^{n!} \nu_\rho = \max_{i,j} (a_{1j} + a_{2j} + \dots + a_{nj};\ a_{i1} + a_{i2} + \dots + a_{in}).$$

König's theorem,[5] also proved by graph-theoretical means, which says that if all of the elements of a matrix are 0 or 1, and if each row and each column contains exactly k elements equal to 1, then the matrix is the sum of exactly k matrices of the type (δ^ρ_{ij}), is obviously a special case of Theorem II.

Finally, the part of conditions (1) and (3) that requires λ_i, μ_j, and ν_ρ, to be integers is dropped, giving immediate extensions of Theorems I and II, valid for square matrices with arbitrary real numbers as elements.

§1

Let (a_{ij}) be a given n by n matrix with elements which are separated into two classes by a property T. Representing the matrix schematically as a grid of n^2 squares, place a mark in the proper square for each element that satisfies T and leave blank those squares that correspond to elements that do not satisfy T.

Further, call any system of lines (rows and columns) that contain all of the marks in the configuration H, obtained as described above, a covering system of lines. Finally, call any subset of the marks of H, that does not contain two marks that fall on the same line, an independent system of marks.

Then König's theorem clearly assumes the following form. For all configurations of marks H (written in a square grid), the minimum number of lines necessary to cover H is equal to the maximum number of independent marks that can be chosen from H.

The theorem is proved by complete induction. Since it is clearly valid for configurations that consist of only one mark, the theorem will be proved

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[5] König, D. Graphok és alkalmazásuk a determinánsok es halmozok eleméletére, Mat. és Term. -tud. Ért. 34. köt. (1916) pp. 104 - 119 and Math. Ann. 77 (1916) pp. 453-65.

in general if, assuming it to be true for all configurations H_N with at most N marks, it can be shown for all configurations H_{N+1} with N + 1 marks.

To this end, we shall show that when one mark is added to an H_N with N marks to complete it to an H_{N+1} with N + 1 marks, the characteristic numbers m and M (which are assumed to be equal for the configuration H_N and which obviously do not decrease with the adjunction of the (N + 1)-st mark) either both remain the same or both increase by one with the adjunction.

The blank squares in the grid (in which the (N +1)-st mark may be written) are separated into two classes with the help of the minimal covering systems for the given configuration H_N: 1. all those that are covered by at least one minimal covering system for H_N, 2. all those that are covered by no minimal covering system for H_N.

In the first place, it is contended that if the (N + 1)-st mark is written in an arbitrary square of the first class, then the numbers m and M do not increase and hence do not change.

On the one hand, it is clear that, since every square from the first class is covered by at least one minimal covering system for H_N with m lines, the configuration H_{N+1} is covered by the same m lines, and hence m is not increased by the adjunction.

On the other hand, since the configuration H_{N+1} can be covered by m lines, following the definition of independent marks (and using the Schubfach Prinzip), not more than m = M independent marks can be chosen from these lines. Thus, for this kind of adjunction, M does not increase.

Further, it is contended that if the (N + 1)-st mark is written in a square of the second class, then the characteristic numbers m and M both increase by one.

On the one hand, it is clear that H_{N+1} cannot be covered by m lines, since such a set of m lines covering H_{N+1} would be a minimal covering system for H_N which covers the square chosen to adjoin, and hence this square would not be contained in the second class. It is obvious that H_{N+1} can be covered by $m + 1$ lines and hence the adjunction adds one to m.

On the other hand, it is contended that it is possible to form a system of $M + 1$ independent marks from a configuration H_N with $m = M$ independent marks in which an $(N + 1)$-st mark has been written in a square of the second class. Thus, the adjunction also adds one to M.

Choose a minimal covering system for H_N, which contains χ rows and $m - \chi$ columns and (as can be arranged easily) order the grid of lines so that the square from the second class in which the $(N + 1)$-st mark will be written lies in the first row and column and so that the χ rows and $m - \chi$ columns of the minimal covering system are the last rows and columns.

Then (see Figure 1) the part of the configuration H_N that falls in the rectangle ABCD has the minimal covering number χ. Namely, if $\chi - 1$ lines would cover this configuration, then these $\chi - 1$ lines plus the first column plus the last $m - \chi$ columns would be a minimal covering system for H_N covering the square that we have chosen to adjoin; however, this means that the square does not belong to the second class, contrary to assumption. Therefore, by the induction assumption, the configuration falling in the rectangle ABCD contains χ independent marks.

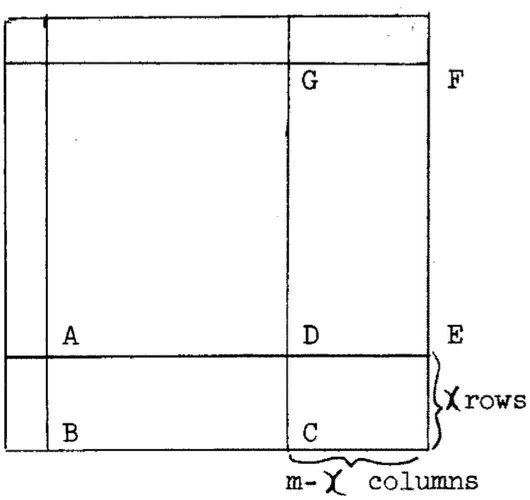

Figure 1

By analogous reasoning, it follows that the configuration falling in DEFG contains $m - \chi$ independent marks. Then, since the former χ marks, the latter $m - \chi$ marks, and the adjoined mark form a system of $m + 1 = M + 1$ independent marks, M is increased by one by the adjunction in this case.

Taking into account the meaning of the formation of the configuration of marks H_N, we obtain the following statement, corresponding to König's original result:

> If the elements of a matrix are divided into two classes by a property T then the minimum number of lines that contain all of the elements with the property T is equal to the maximum number of elements with the property T, with no two on the same line.

Let (a_{ij}) be an n by n [§ 2] matrix with non-negative integers as elements. A system of lines that contains the i-th row with the multiplicty λ_i and the j-th column with the multiplicity μ_j is called a covering system of lines if, for all values of i and j :

(1) $$\lambda_i + \mu_j \geq a_{ij} \qquad (i, j = 1, 2, \ldots, n).$$

A covering system of lines that contains a minimum number of lines, $\sum_k(\lambda_k + \mu_k)$, is called a minimal covering system. Further, the following sums are called diagonal sums:

(5) $$a_{1\nu_1} + a_{2\nu_2} + \ldots . + a_{n\nu_n},$$

where $\nu_1, \nu_2, \ldots, \nu_n$ runs through all permutations of 1, 2, ..., n. For these notions defined in this manner we have:

> THEOREM I. If (a_{ij}) is an n by n matrix with non-negative integers as elements, then the minimum number of lines needed to cover it is equal to the maximum diagonal sum; that is, subject to the condition
>
> (1) $\lambda_i + \mu_j \geq a_{ij}$ $(i, j = 1, 2, \ldots, n)$
>
> we have
>
> (2) $\min \sum_{k=1}^{n} (\lambda_k + \mu_k) = \max(a_{1\nu_1} + a_{2\nu_2} + \ldots + a_{n\nu_n}).$

Choose a minimal covering system with multiplicities λ_k^* and μ_k^* for $k = 1, 2, \ldots, n$. Then according to (1),

(6) $\min \sum_{k=1}^{n} (\lambda_k + \mu_k) = \sum_{k=1}^{n} (\lambda_k^* + \mu_k^*) \geq a_{1v_1} + a_{2v_2} + \dots + a_{nv_n}$ for all diagonal sums.

The chosen minimal covering system λ_k^*, μ_k^* serves as a means to divide the elements of a matrix (a_{ij}) into two classes: 1. we call essential those elements a_{pq}^* that satisfy

(7) $\lambda_p^* + \mu_q^* = a_{pq}^*$

2. we call inessential those elements a_{rs} that satisfy

(7') $\lambda_r^* + \mu_s^* > a_{rs}$.

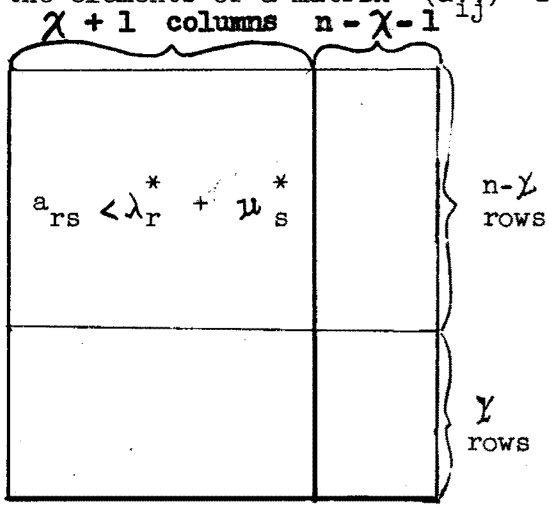

Figure 2

It is contended that the minimum number of lines needed to cover all of the essential elements a_{pq}^* is equal to n. Suppose, to the contrary, that $n - 1$ lines contain all of the essential elements and (as can be arranged easily) that those lines that contain the essential elements are the last χ rows and $n - \chi - 1$ columns. Then the elements common to the first $n - \chi$ rows and first $\chi + 1$ columns are all inessential. That is,

$$\lambda_r^* + \mu_s^* > a_{rs}$$

and thus

$$\lambda_r^* + \mu_s^* \geq 1$$

and so, by necessity,[6] at least one of the systems of inequalities

(8) $\lambda_1^* \geq 1, \lambda_2^* \geq 1, \dots, \lambda_{n-\chi}^* \geq 1$

or

(8') $\mu_1^* \geq 1, \mu_2^* \geq 1, \dots, \mu_{\chi+1}^* \geq 1$

is valid.

If, for instance, all of the inequalities of (8) hold then the system of lines with multiplicities λ_k^{**}, μ_k^{**} defined by

[6] Namely, if neither system of inequalities is valid then there must be at least one r and s, for which $\lambda_r^* = 0$ and $\mu_s^* = 0$, hence $\lambda_r^* + \mu_s^* = 0$, contrary to (7').

$$\overset{**}{\lambda}_r = \overset{*}{\lambda}_r - 1 \qquad (r = 1, 2, \ldots, n-\chi)$$

$$\overset{**}{\lambda}_p = \overset{*}{\lambda}_p \qquad (p = n-\chi+1, \ldots, n)$$

$$\overset{**}{\mu}_s = \overset{*}{\mu}_s \qquad (s = 1, 2, \ldots, \chi+1)$$

$$\overset{**}{\mu}_q = \overset{*}{\mu}_q + 1 \qquad (q = \chi+2, \chi+3, \ldots, n)$$

is obviously a covering system. Also,

$$\sum_{k=1}^{n} (\overset{**}{\lambda}_k + \overset{**}{\mu}_k) = \sum_{k=1}^{n} (\overset{*}{\lambda}_k + \overset{*}{\mu}_k) - 1,$$

and hence the system of lines with multiplicities $\overset{*}{\lambda}_k$, $\overset{*}{\mu}_k$ was not minimal.

Thus, we have proved that the minimal number of lines needed to contain all of the essential elements $\overset{*}{a}_{pq}$ is equal to n. According to the theorem proved in §1 we can choose n essential elements $\overset{*}{a}_{1q_1}$, $\overset{*}{a}_{2q_2}$, ..., $\overset{*}{a}_{nq_n}$ so that no two lie on the same line. Forming a diagonal sum with these and taking account of equations (7), we have:

$$(9) \quad \overset{*}{a}_{1q_1} + \overset{*}{a}_{2q_2} + \ldots + \overset{*}{a}_{nq_n} = \sum_{k=1}^{n} (\overset{*}{\lambda}_k + \overset{*}{\mu}_{q_k}) = \sum_{k=1}^{n} (\overset{*}{\lambda}_k + \overset{*}{\mu}_k).$$

Finally, on the basis of (6) and (9),

$$(2) \qquad \min \sum_{k=1}^{n} (\lambda_k + \mu_k) = \max (a_{1\nu_1} + a_{2\nu_2} + \ldots + a_{n\nu_n}) \quad \text{Q. E. D.}$$

If a matrix (a_{ij}) is given with non-negative rational numbers as elements and the least common denominator A of the elements a_{ij} is used to write them in the form $\frac{\alpha_{ij}}{A}$ (where the α_{ij} are non-negative integers), then Theorem I, applied to the matrix (α_{ij}), yields the following:

$$\min \sum_{k=1}^{n} \left(\frac{\lambda_k}{A} + \frac{\mu_k}{A} \right) = \max \sum_{k=1}^{n} \left(\frac{\alpha_{k\nu_k}}{A} \right) = \max \sum_{k=1}^{n} a_{k\nu_k} .$$

In this case, the numbers λ_k and μ_k in the relations (1) (which are considered as integral multiplicities of lines when the elements of the matrix are integers) may be considered as weights in a covering system and the theorem expressed by means of the relations (2) remains valid if rational or --- upon simple considerations of continuity --- arbitrary real non-negative numbers are taken as elements of the matrix.

§3

Our treatment of Theorem II makes use of the following terminology. Let the n! distinct n by n matrices (δ^{ρ}_{ij}) with $\rho = 1, 2, \ldots, n!$ that arise from the unit matrix through permutations of the rows (or columns) be called diagonal lines (in contrast to the "parallel" lines, applying to either rows or columns). A system of diagonal lines which contains the diagonal line (δ^{ρ}_{ij}) with multiplicity ν_ρ is called a diagonal covering system if, for all values of i and j,

(3) $$\sum_{\rho=1}^{n!} \nu_\rho \delta^{\rho}_{ij} \geq a_{ij} \quad (i, j = 1, 2, \ldots, n).$$

The diagonal covering systems that make the diagonal sum: $\sum_{\rho=1}^{n!} \nu_\rho$ a minimum, are called minimal diagonal covering systems. The sums

(10) $$a_{i1} + a_{i2} + \ldots + a_{in} \quad (i = 1, 2, \ldots, n)$$
$$a_{1j} + a_{2j} + \ldots + a_{nj} \quad (j = 1, 2, \ldots, n)$$

are called parallel sums (in contrast to the diagonal sums used previously). Then the following theorem is valid:

THEOREM II. If (a_{ij}) is an n by n matrix with non-negative integers as elements, then the minimum number of diagonal lines needed to cover it is equal to the maximum parallel sum; that is, subject to the condition

(3) $$\sum_{\rho=1}^{n!} \nu_\rho \delta^{\rho}_{ij} \geq a_{ij} \quad (i, j = 1, 2, \ldots, n)$$

we have

(4) $$\min \sum_{\rho=1}^{n!} \nu_\rho = \max_{i,j} (a_{1j} + a_{2j} + \ldots a_{nj}; a_{i1} + a_{i2} + \ldots + a_{in}).$$

The proof of this theorem makes use of a line of thought originating with König,[7] according to which it is sufficient to prove the theorem for matrices which have all parallel sums equal. Namely, we shall define, for all (a_{ij}), a "majorant" (a^*_{ij}) such that

(11) $$a^*_{ij} \geq a_{ij} \quad (i, j = 1, 2, \ldots, n)$$

and

(11') $$a^*_{p1} + a^*_{p2} + \ldots + a^*_{pn} = a^*_{1q} + a^*_{2q} + \ldots + a^*_{nq} = M =$$
$$= \max_{i,j}(a_{i1} + a_{i2} + \ldots + a_{in};\ a_{1j} + a_{2j} + \ldots + a_{nj}),\ (p, q = 1, 2, \ldots, n)$$

thus which has no element less than the corresponding element of the given matrix and which has all parallel sums equal to the maximum parallel sum M for the matrix (a_{ij}).

Namely, if the matrix (a_{ij}) does not have all parallel sums equal, then it must contain a row p and a column q with parallel sums less than the maximum parallel sum M:[8]

$$a_{p1} + a_{p2} + \ldots + a_{pn} < M \text{ and } a_{1q} + a_{2q} + \ldots + a_{nq} < M.$$

Then substitute for the element a_{pq} common to the pth row and qth column

$$a_{pq} + M - \max(a_{p1} + \ldots + a_{pn},\ a_{1q} + \ldots + a_{nq}).$$

This obviously yields a majorizing matrix with at least one more parallel sum equal to M than in the original matrix. Therefore, after no more than $2n - 1$ iterations, this procedure leads to a majorant (a^*_{ij}) for the original matrix (a_{ij}) which has all parallel sums equal to M.

Since all covering systems for the majorant (a^*_{ij}) are a fortiori covering systems for the original matrix (a_{ij}), it is sufficient to prove the theorem for the majorant (a^*_{ij}).

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[7] l. c. [5]), III.

[8] Namely, if all row sums are equal to M, then all column sums are equal to M, and vice versa.

The theorem is obviously true for all matrices (a^*_{ij}) which have all parallel sums equal to one (thus, in which all rows and columns contain exactly one element equal to one). If therefore, it can be deduced --- assuming the theorem valid for all matrices with all parallel sums equal to $M - 1$ --- for all matrices with all parallel sums equal to m, then the theorem is proved for the general case.

To begin with, the minimum number of (parallel) lines needed to contain all elements of the matrix (a^*_{ij}) which are different from zero, is equal to n. On the one hand, the sum of all elements is equal to $n.M$ and, on the other hand, all the parallel sums are equal to M. Hence no fewer than n (parallel) lines can contain all of the non-zero elements. Thus, according to the theorem of 1, one can choose n distinct non-zero elements $a^*_{1\nu_1}$, $a^*_{n\nu_n}$ with no two on the same (parallel) line.

If $(\delta^{\rho *}_{ij})$ is the diagonal line defined by

$$(12) \qquad \delta^{\rho *}_{ij} = \begin{cases} 1, & \text{if } j = \nu_i \\ 0, & \text{if } j \neq \nu_i \end{cases} \quad (i, j = 1, 2, \ldots, n)$$

then $(a^*_{ij} - \delta^{\rho *}_{ij})$ is a matrix with non-negative integers for elements, which has all parallel sums equal to $M - 1$, and thus, by assumption, is covered by $M - 1$ diagonal lines. Consequently, these $M - 1$ diagonal lines plus $(\delta^{\rho *}_{ij})$ cover (a^*_{ij}) and, a fortiori, these M lines together cover the given (a_{ij}), which was majorized by (a^*_{ij}).

If the maximum parallel sum for a matrix (a_{ij}) is M, then obviously no fewer than M diagonal lines can cover (a_{ij}), since each diagonal line contains exactly one element in each parallel line. Therefore

$$(4) \qquad \min \sum_{\rho=1}^{n!} \gamma_\rho = M = \max\left(\sum_{k=1}^{n} a_{ik}, \sum_{k=1}^{n} a_{kj}\right). \qquad \text{Q. E. D.}$$

Theorem II, like Theorem I, can be extended to matrices with non-negative rational and real numbers as elements.

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