TWO THEOREMS IN GRAPH THEORY

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Communicated by N. E. Steenrod, July 8, 1957

Introduction.—Given an unoriented graph (or 1-dimensional regular complex), let X be the set of all its vertices and U be the set of all its edges. When the graph is finite, the following problems arise:

Problem 1: A set $A \subset X$ is said to be *internally stable* if $x \in A$, $y \in A$ implies $(x, y) \notin U$. The symbol $|A|$ will denote the number of elements of A. Construct an internally stable set A such that $|A|$ is maximum.

Problem 2: A set $B \subset X$ is said to be a *cover* if every edge of U is adjacent to at least one vertex in B. Construct a cover with the minimum number of elements.

Problem 3: A set of edges $V \subset U$ is said to be a *matching* if two edges of V have no vertex in common. Construct a matching with the maximum number of elements.

A particular case of Problem 1 is the chess problem of Gauss: Put eight queens on the board such that no one can take any other. In n-person game theory, if the graph of domination is symmetrical, a maximum internally stable set turns out to be a maximum *solution* (in the von Neumann-Morgenstern sense[1]), and the more usual case can be solved by means of the Grundy functions.[2]

Problem 2 is the set theoretic dual of Problem 1, since the complement of an internally stable set is a cover, and conversely. Particular cases of Problem 3 are the problem of distinct representatives (P. Hall[3]) and the problem of Petersen (D. König[4]). In the case where the graph is bipartite, Problem 3 has been solved by algebraic methods by O. Ore,[5] and an efficient algorithm has been given by H. Kuhn.[6] Unfortunately, the linear programming duality used by H. Kuhn no longer subsists when the graph is not bipartite. (Note that Problem 2 is the linear program dual to Problem 3 in the bipartite case.) In view of solving the general case, this paper states two theorems: Theorem 1 gives a necessary and sufficient condition for recognizing whether a matching is maximum and provides an algorithm for Problem 3, while Theorem 2 yields an algorithm for Problems 1 and 2.

The Theorems.—Consider a graph $G = (X, U)$ with a matching V_0; if $u \in V_0$ we shall say that edge u is *strong*, otherwise that u is *weak*. An *alternating chain* is a chain which does not use the same edge twice and is such that for any two adjacent edges one is strong and the other is weak. A vertex x which is not adjacent to a strong edge is said to be *neutral*, the set of all neutral points being N.

We shall also consider a graph $\bar{G}$ constructed from G by adding a vertex $\bar{a}$ and connecting $\bar{a}$ to every neutral point with a strong edge. If there exists an alternating chain from $\bar{a}$ to a vertex x, we shall picture an arrow on the last edge (z, x), directed from z to x. A vertex x ($\notin N$) which is not adjacent to a directed edge is said to be *inaccessible*, the set of all inaccessible points being I. A vertex x ($\notin N$) adjacent to a weak edge directed to x and not to a strong edge directed to x is said to be *weak*, the set of all weak points being W. A vertex x ($\notin N$) adjacent to a strong edge directed to x and not to a weak edge directed to x is said to be *strong*, the set of all strong points being S. A vertex x ($\notin N$) adjacent to a strong edge directed to x and to a weak edge directed to x is said to be *medium*, and the set of all medium points will be designated by M.

LEMMA 1. *Let Y be a connected component of the subgraph M; if $\bar{a}$ is inaccessible, there exists in $\bar{G}$ one strong edge adjacent to Y and directed to Y only; all other edges adjacent to Y are weak and directed from Y only. Moreover, all vertices not in Y and connected to Y by one edge are weak, and $|Y| \geq 3$.*

This is a theorem of T. Gallai;[7] a shorter proof is given by Berge.[8]

LEMMA 2. *If $\bar{a}$ is inaccessible, $S \cup N$ is internally stable.*

(Immediate.)

LEMMA 3. *If $\bar{a}$ is inaccessible, $M = \phi$ and $I = \phi$, then $S \cup N$ is a maximum internally stable set, W is a minimum cover, and V_0 is a maximum matching.*

From Lemma 2, $S \cup N$ is internally stable, hence $W = X - (S \cup N)$ is a cover. For every cover C and for every matching V, one has $|C| \geq |V|$; as $|W| = |V_0|$, the cover W is minimum and the matching V_0 is maximum.

LEMMA 4. *Let Z be a connected component of the subgraph I; if $\bar{a}$ is inaccessible, all edges adjacent to Z are weak and undirected; moreover, all vertices not in Z connected to Z by an edge are weak, and $|Z| \geq 2$.*

(Immediate.)

LEMMA 5. *If $|N| \leq 1$, V_0 is a maximum matching.*

This follows from the fact that $|X| = 2|V_0| + |N|$.

LEMMA 6. *If $A \subset X$, let G_A be the graph constructed from G by shrinking A into a single vertex a_A, having as adjacent edges the adjacent edges of A. If the original strong edges constitute a maximum matching for the subgraph A, and for G_A, then V_0 is a maximum matching for G.*

This is easy to see by an induction on the number of elements of A.

THEOREM 1. *A matching V is maximum if and only if there does not exist an alternating chain connecting a neutral point to another neutral point.*

If there existed an alternating chain $W = (u_1, u_2 \ldots, u_k)$ connecting a neutral point a to a neutral point a' different from a, $(V - W) \cup (W - V)$ would be a matching with more elements than V, and V would not be maximum.

Conversely, let us prove that, if such a chain does not exist, V is maximum; the proposition being obvious when the graph has one or two edges, we shall assume that the proposition is true for any graph having fewer than m edges, and we shall prove it for a graph G of m edges. One can assume that G is connected.

From Lemma 5, one can assume $|N| > 1$; from Lemma 3, one can also assume that either $M \neq \phi$ or $I \neq \phi$.

1. If $M \neq \phi$, let Y be a connected component of the subgraph M; the graph G_Y constructed from G by shrinkage satisfies the conditions of the theorem (Lemma 1); as it has at least one edge less than G, the strong edges constitute a maximum matching for G_Y. On the other hand, the subgraph Y has only one neutral point (Lemma 1) and therefore its strong edges constitute a maximum matching. Thus, from Lemma 6, V_0 is a maximum matching for G.

2. If $I \neq \phi$, let Z be a connected component of subgraph I, and consider the graph G_Z. The vertex a_Z is a neutral point, connected only with weak points. No alternating chain leads from a point of N to a_Z. As G_Z satisfies the conditions of the theorem, G_Z admits its strong edges as a maximum matching. On the

other hand, the subgraph Z, having no neutral points, admits its strong edges as a maximum matching; therefore, V_0 is a maximum matching for G.

THEOREM 2. *Let C_Y (resp. C_Z) be any minimum cover for the subgraph generated by a connected component Y of M (resp. Z of I). If there does not exist an alternating chain connecting a neutral point to another neutral point, the set*

$$C = W \cup \bigcup_Y C_Y \cup \bigcup_Z C_Z$$

is a minimum cover for G.[9]

Every vertex which is connected by an edge to a component Y is a weak point (Lemma 1); every vertex which is connected by an edge to a component Z is a weak point (Lemma 4). Therefore C is a cover for G. As C is a minimum cover for the graph G' deduced from G by removing all edges connecting a weak vertex to a medium or inaccessible vertex (Lemma 3), C is also a minimum cover for G.

Theorem 1 suggests the following procedure for solving Problem 3; Construct a maximal matching V, and determine whether there exists an alternating chain W connecting two neutral points. (The procedure is known.) If such a chain exists, change V into $(V - W) \cup (W - V)$, and look again for a new alternating chain; if such a chain does not exist, V is maximum.

Theorem 2 gives an algorithm for Problem 2, hence for Problem 1.

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[6] H. Kuhn, "The Hungarian Method for the Assignment Problem," *Naval Research Logistics Quart.*, **2**, 83–97, 1955.

[7] T. Gallai, "On Foundation of Graphs," *Acta Math. Hung.*, **1**, 133–153, 1950.

[8] C. Berge, *Théorie des Graphes* (Dunod publ., in preparation).

[9] R. Z. Norman and Michael O. Rabin proved independently a similar theorem (cf. An algorithm for a minimum cover of a graph, [Abstract], Washington, D. C. meeting of the A.M.S., October 26, 1957) which could be in some sense a dual of this result and which yields an algorithm for the following problem: construct a set C of edges such that every vertex is incident to an edge of C, and which has a minimum number of edges.