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\footnote{The slides were made based on Chapter 13 of Algorithm design, Randomized Algorithm by R. Motwani and P. Raghavan.}
Introduction and nine categories proposed by R. Karp;
The first example: GlobalMinCut problem;
Randomized algorithm in protocol design for distributed system;
Randomization in approximation algorithm: LP+Random Rounding;
Randomization coupled with divide-and-conquerer;
Hashing

Why randomized algorithm? Simplicity and speed. For many applications, a randomized algorithm is the simplest algorithm available, the fastest, or both.
A brief introduction
How to deal with hard problems? Trade-off “quality” and “time”

We have a couple of options:

1. **Give up polynomial-time** restriction: hope that our algorithms run fast on the practical instances. (e.g. branch-and-bound, branch-and-cut, and branch-and-pricing algorithms are used to solve a TSP instance with over 24978 Swedish Cities. See http://www.tsp.gatech.edu/history/pictorial/sw24978.html)

2. **Give up optimum** restriction: from “optimal” solution to “nearly optimal” solution in the hope that “nearly optimal” is easy to find. e.g., approximation algorithm (with theoretical guarantee), heuristics, local search (without theoretical guarantee);

3. **Give up deterministic** restriction: the expectation of running time of a randomized algorithm might be polynomial;

4. **Give up worst-case** restriction: algorithm might be fast on special and limited cases;
Goal: To prove that the algorithm solves the problem correctly (always) and quickly (typically, the number of steps should be polynomial in the size of the input)

(Excerpted from slides by P. Raghavan)
Randomized algorithm

- In addition to input, algorithm takes a source of random numbers and makes random choices during execution.
- Behavior can vary even on a fixed input
The world is random: our algorithm is a deterministic algorithm that confront randomly generated input, and we can study the behavior of an algorithm on an “average” input rather than the worst-case input.

The algorithm is random: the world provides the same worst-case input as always; however, we allow our algorithm to make random decisions during execution.
1 Las Vegas: always correct. Analyze its expected running-time.
2 Monte Carlo: correctness depends on the random choice. Analyze its error probability.

Note: still for worst-case input. \( \max_{\text{Instance}} \text{expected time, or } \max_{\text{Instance}} \Pr[\text{error}] \);
Paradigms for randomized algorithms
A handful of general principles lies at the heart of almost all randomized algorithms, despite the multitude of areas in which they find application.

1. Foiling an adversary: The classical adversary argument for a deterministic algorithm establishes a lower bound on the running time by constructing an input on which the algo fares poorly. While the adversary may be able to construct an input to foil one deterministic algo, it is difficult to devise a single input that is likely to defeat a randomized algo. (online algo, efficient proof verification)

2. Random sampling: a random sample from a population is representative of the whole population;
Abundance of witnesses: To find a witness of a property of an input (say, “$x$ is prime”). If the witness lies at a search space that is too large to search exhaustively, it suffices to randomly choose an element if the space contains a large number of witnesses.

Fingerprinting and hashing: to represent a long string by a short fingerprint using a random mapping. If two fingerprints are identical, the two strings are likely to be identical.

Random re-ordering input: After the re-ordering step, the input is unlikely to be in one of the orderings that is pathological for the naive algorithm;
Rapidly mixing Markov chains: To count the number of combinatorial objects, we can randomly sample an appropriately defined population, which in turn relies on the information of the number. Solution: defining a Markov chain on the elements, and showing a short random walk is likely to sample the population uniformly.

Probabilistic methods and existence proofs: To establish that an object with certain property exists by arguing that a randomly chosen object has the property with positive probability.

Load balancing: To spread load evenly among the resources in a parallel or distributed environment, where resource utilization decisions have to be made locally without reference to the global impact of these decisions. To reduce the amount of explicit communication or synchronization.
Isolation and symmetry breaking: In parallel environment, it is usually important to require a set of processors to find the same solution (consensus): choosing a random ordering on the feasible solutions, and then requiring the processors to find the solution with the lowest rank.
The first example: **GLOBALMINCUT** problem
**Graph algorithm: ** **GLOBAL-MIN-CUT** problem

**INPUT:**
A graph $G = \langle V, E \rangle$

**OUTPUT:** a cut $c = \langle A, B \rangle$ such that the size of $c$ is minimized. Here, $A, B$ are non-empty vertex sets and $V = A \cup B$.

Note: The difference from the $s-t$ cut problem, where two vertex $s$ and $t$ are given, and we restrict the cut: $s \in A$, and $t \in B$. 

![Diagram](https://via.placeholder.com/150)
Basic idea: Transferring undirected graph $G$ to a directed graph $G'$ by replacing an edge $e = (u, v)$ with $e' = (u, v)$ and $e'' = (v, u)$, each of capacity 1. Let $s$ be an arbitrary node. For $t = 2$ to $n$, call maximum-flow algorithm to calculate the minimum $s - t$ cut, and report the minimal one.

Intuition: If $(A, B)$ is a minimum $s - t$ cut in $G'$, $(A, B)$ is also a minimum cut among all those that separates $s$ from $t$. We need a cut to separate $s$ from something.

Time-complexity: $O(n^4)$.

Note: Global minimum cut can be found as efficiently as $s - t$ cut by techniques that didn’t requires augmentation-path or a notion of flow.
Randomized algorithm (D. Karger, ’92)

Advantage: qualitatively simpler than all previous algorithms.

Contraction Algo( G )
Initially, S(v) = \{v\};
//S(v) to recorder nodes that have been contracted to v;

If G have two nodes v1 and v2
then return S(v1) and S(v2);

Choose an edge e=<u,v> uniformly at random;
Contract G to G', where a new node Zuv replace u and v;
Remove all edge between u and v;
S(Zuv) = S(u) + S(v);
Contraction( G' );
A Las Vegas algorithm for **GLOBAL MIN CUT** problem
The contraction algorithm returns a global min-cut with probability at least $\frac{1}{C_2^n}$.

Note: a bit counter-intuitive since there are exponentially many possible cuts of $G$, and thus the probability seems to be exponentially small.

Proof:

- Suppose $(A, B)$ is a global min-cut with $k$ edges. We want a lower bound of the probability that Contraction algo returns $(A, B)$.

- Complement: failure due to an edge $e = (u, v), u \in A, v \in B$ is selected for contracting;

- Let $F_i$ be the event that an edge $e$ in the cut is selected at the $i$-th iteration. We have:
(The 1-st iteration) $Pr[F_1] = \frac{k}{|E|} \leq \frac{k}{(1/2)kn} = \frac{2}{n}$.
(Reason: Each node $v$ has a degree at least $k$. Otherwise, the cut $(v, V - v)$ has a size less than $k$. Thus, the edge number is at least $(1/2)kn$.)

(The $j$-th iteration) $Pr[F_j | F_{j-1}...F_1] \leq \frac{2}{n-j}$.
(Reason: same argument to $G'$, where only $n - j$ supernodes are left.)

\[
Pr[F_{n-2} \cap ... \cap F_1] = Pr[F_1] Pr[F_2 | F_1]...Pr[F_{n-2} | F_{n-3} \cap ... \cap F_1] \tag{2}
\]
\[
\geq (1 - \frac{2}{n})(1 - \frac{2}{n-1})...(1 - \frac{2}{3}) \tag{3}
\]
\[
= \frac{n - 2}{n} \cdot \frac{n - 3}{n - 1} \cdot \frac{n - 4}{n - 2} \cdot ... \cdot \frac{1}{3} \tag{4}
\]
\[
= \frac{2}{n(n-1)} \tag{5}
\]
Further reduce failure probability via repeating

Basic idea: running Contraction algo $r$ times will increase the probability to find a global min-cut.

- $r = C_n^2$: $Pr(FAILURE) \leq (1 - \frac{1}{C_n^2})^{C_n^2} \leq \frac{1}{e}$.
- $r = C_n^2 \ln n$: $Pr(FAILURE) \leq (1 - \frac{1}{C_n^2})^{C_n^2 \ln n} \leq \frac{1}{e^{\ln n}} = \frac{1}{n}$.

Time complexity: $O(rm)$ (Contraction algo costs $O(m)$ time.)
Question: what is the maximum number of global min-cuts an undirected graph $G$ can have?

Not obvious. Consider a directed graph as follows: $s$ together with any subset of $v_1, ..., v_n$ constitutes a minimum $s - t$ cut. ($2^n$ cuts in total.)
Theorem

An undirected graph \( G \) on \( n \) nodes has at most \( \binom{n}{2} \) global min-cuts.

Proof:

- Suppose there are \( r \) global min-cut \( c_1, \ldots, c_r \);
- Let \( C_i \) denote the event that \( c_i \) is reported, and \( C \) denote the success of Contraction algo;
- For each \( i \), we have \( \Pr[C_i] \geq \frac{1}{\binom{n}{2}} \).
- Thus \( \Pr[C] = \Pr[C_1 \cup \ldots \cup C_r] = \sum_i \Pr[C_i] \geq r \frac{1}{\binom{n}{2}} \). (Note: \( = \) since all \( C_i \) are disjoint. )
- We get \( r \leq \binom{n}{2} \). (\( r \frac{1}{\binom{n}{2}} \leq 1 \).)
Randomization in distributed computing
INPUT:
Suppose we have \( n \) nodes \( M_1, ..., M_n \), each competing for access to a single shared database. The database can be accessed by at most one node in a single time slice; if two or more nodes attempt to access it simultaneously, then all nodes are “locked out” for the duration of that slice.

GOAL:
to design a protocol to divide up the time slices among the nodes in an equitable fashion. (suppose that the nodes cannot communicate with one another at all.)
Protocol design: Contention Resolution

- A randomized algorithm: *Each node will attempt to access the database at each slice with probability $p$, independently of the decisions of others.*
- Intuition: each node randomizes its behavior.
- Symmetry-breaking strategy: If all nodes operated in lockstep, repeatedly trying to access the database at the same time, there’d be no progress; but by randomizing, they “smooth out” the contention.
Waiting for a particular node to succeed.

**Theorem**

After $\Theta(n)$ time slices, that probability that $M_i$ has not yet succeeded is less than a constant; and after $\Theta(n \ln n)$ time slices, the probability drops to a quite small quantity.
Proof.

Let \( A(i, t) \) denote the event that \( M_i \) attempts to access DB at time \( t \), and \( S(i, t) \) denote the success of the access.

We have:
\[
Pr[S(i, t)] = Pr[A(i, t)] \times \prod_{j \neq i} Pr[A(j, t)] = p(1 - p)^{n-1}
\]

By setting the derivative to 0, we get \( p = \frac{1}{n} \). And the maximum of \( Pr[S(i, t)] \) is achieved: \( Pr[S(i, t)] = \frac{1}{n}(1 - \frac{1}{n})^{n-1} \).

\[
\frac{1}{en} \leq Pr[S(i, t)] \leq \frac{1}{2n}.
\]
(Reason: As \( n \) increases from 2, \( (1 - \frac{1}{n})^n \) converges monotonically from \( \frac{1}{4} \) to \( \frac{1}{e} \), and \( (1 - \frac{1}{n})^{n-1} \) converges monotonically from \( \frac{1}{2} \) to \( \frac{1}{e} \).)

Let \( F(i, t) \) denote the “failure event” that \( P_i \) does not succeed in any of the slices 1 through \( t \);

\[
Pr[F(i, t)] = \prod_{r=1}^{t} Pr[S(i, r)] = (1 - \frac{1}{n}(1 - \frac{1}{n})^{n-1})^t.
\]

A simpler estimation: \( Pr[F(i, t)] = \prod_{r=1}^{t} Pr[S(i, r)] \leq (1 - \frac{1}{en})^t. \)

\[
Pr[F(i, t)] \leq (1 - \frac{1}{en})^t \leq \frac{1}{e} \text{ when setting } t \text{ to } en.
\]

\[
Pr[F(i, t)] \leq (1 - \frac{1}{en})^t \leq (\frac{1}{e})^{c\ln n} = n^{-c} \text{ when setting } t \text{ to } cen \ln n.
\]
Analysis of the protocol

Waiting for all nodes to succeed.

Theorem

With probability at least $1 - n^{-1}$, all nodes succeed in accessing the DB at least once within $t = 2en \ln n$ time slices.

Proof.

- Let $F(t)$ denotes the event that some nodes have not yet accessed DB after $t$ time slices;
- $Pr[F(t)] = Pr[\bigcup_{i=1}^{n} F(i, t)] \leq \sum_{i=1}^{n} Pr[F(i, t)]$
- $Pr[F(t)] \leq n \times n^{-c}$ after $t = cen \ln n$ time slices.
- In particular, $Pr[F(t)] \leq \frac{1}{n}$ after $t = 2en \ln n$ time slices.
Protocol design in distributed system: LoadBalance

**INPUT:** $n$ processors $P_1, \ldots, P_n$, and $m$ jobs arrive in a stream and need to be processed immediately;

**GOAL:** to design a protocol to distribute jobs among processors evenly. (Assuming no central controller again.)

Randomized algorithm: assign each job to one of the processors uniformly at random.
Analysis: how well does this simple algo work?

Theorem

(A simple case: $m = n$) With probability at least $1 - n^{-1}$, there is no processor that was assigned with over $e\gamma(n) = \Theta\left(\frac{\log n}{\log \log n}\right)$ jobs.
Proof.

- Let $X_i$ denote the number of jobs assigned to $P_i$. Define an index random variable $Y_{ij}$ as follows: $Y_{ij} = 1$ when job $j$ is assigned to $P_i$, and 0 otherwise.

- We have: $X_i = \sum_{j=1}^{n} Y_{ij}$.

- Then,
  \[
  E(X_i) = E(\sum_{j=1}^{n} Y_{ij}) \\
  = \sum_{j=1}^{n} E(Y_{ij}) \\
  = \sum_{j=1}^{n} Pr(Y_{ij} = 1) \\
  = n \times \frac{1}{n} = 1
  \]

- Thus $Pr[X_i > c] < \frac{e^{c-1}}{cc}$ (by Chernoff bound.)

- Suppose we have a $c$ such that $Pr[X_i > c] < \frac{e^{c-1}}{cc} \leq \frac{1}{n^2}$, then $Pr[\exists i, X_i > c] \leq n \times \frac{1}{n^2} = \frac{1}{n}$. 


The remaining difficulty: how to choose a $c$?

- Let $\gamma(n)$ be the solution to $x^x = n$. The estimation of $\gamma(n)$ can be given as follows:
- Taking logarithm of $x^x = n$ gives: $x \ln x = \ln n$.
- Taking logarithm again: $\ln x + \ln \ln x = \ln \ln n$.
- We have: $\ln x \leq \ln \ln n = \log x + \ln \ln x \leq 2 \ln x$ (by $\log(\log(x)) \leq \log(x)$) Dividing the equation: $x \ln x = \ln n$ (by $\ln \ln n(n) \geq 0$ when $n \geq e^e$), we get:
- $\frac{1}{2}x \leq \frac{\ln n}{\ln \ln n} \leq x = \gamma(n)$.
- Setting $c = e\gamma(n)$, we have:
- $Pr[X_i > c] < \frac{e^{c-1}}{c c} < \left(\frac{e}{c}\right)^c = \left(\frac{1}{\gamma(n)}\right)^{e\gamma(n)} < \left(\frac{1}{\gamma(n)}\right)^{2\gamma(n)} = \frac{1}{n^2}$. 
$\gamma(n)$: the solution to $x^x = n$
More jobs: \((m = 6n \ln n)\) The expected jobs number is: \(\mu = 6 \ln n\). We have:

\(Pr[X_i > 2\mu] < \left(\frac{e}{4}\right)^6 \ln n < \left(\frac{1}{e^2}\right)^\ln n = \frac{1}{n^2}. \) (by \(\left(\frac{e}{4}\right)^6 < \frac{1}{e^2}.\))
Bounding the sum of independent random variables
Bound 1: Markov inequality

Suppose $X$ is a non-negative random variable with mean $u = E(X)$.

We have: $\Pr[X \geq t] \leq \frac{E(X)}{t}$.
Suppose $X$ is a random variable with mean $u = E(X)$, and variance $\sigma^2 = Var(X)$.

We have: $Pr[|X - u| \geq k\sigma] \leq \frac{1}{k^2}$.

Note: the non-negative requirement is removed, but need the information of variance.
(Upper bound) Let $X = X_1 + X_2 + \ldots + X_n$, where $X_i$ is a 0/1 variable that takes 1 with probability $p_i$. Define $\mu = E(X) = \sum_i p_i$. For any $\delta > 0$, we have:

$$Pr[X > (1 + \delta)\mu] < \left(\frac{e^\delta}{(1+\delta)(1+\delta)}\right)^\mu.$$ 

(Intuition: the fluctuations of $X_i$ are likely to be “cancelled out” as $n$ increases.)
Proof.

1. **Step 1:** 
   \[ Pr[X > (1 + \delta)\mu] = Pr[tX > t(1 + \delta)\mu] = Pr[e^{tX} > e^{t(1+\delta)\mu}] \leq \frac{E(e^{tX})}{e^{t(1+\delta)\mu}} \text{ for any } t > 0. \]  
   (Applying Markov inequality on \( e^{tX} \))

2. **Step 2:** 
   \[ E(e^{tX}) = E(e^{tX_1 + \ldots + tX_n}) = E(e^{tX_1} \ldots e^{tX_n}) \] (by independence of \( X_i \)).

3. **Step 3:** 
   \[ E(e^{tX_i}) = e^tp_i + 1(1 - p_i) = 1 + p_i(e^t - 1) \leq e^{p_i}(e^t - 1) \] (by \( 1 + x \leq e^x \), for \( x > 0 \)).

4. **Step 4:** Setting \( t = \ln(1 + \delta) \).

Thus we have:

\[ Pr[X > (1 + \delta)\mu] \leq \frac{E(e^{tX})}{e^{t(1+\delta)\mu}} \leq \prod_i e^{p_i(e^t-1)} = \frac{e^{\mu(e^t-1)}}{e^{t(1+\delta)\mu}}. \]
Theorem

(Lower bound) Let $X = X_1 + X_2 + \ldots + X_n$, where $X_i$ is a 0/1 variable that takes 1 with probability $p_i$. Define

$\mu = E(X) = \sum_i p_i$. For any $\delta > 0$, we have:

$Pr[X < (1 - \delta)\mu] < e^{-\frac{1}{2}\mu\delta^2}$. 
Comparison of Markov inequality and Chernoff bound

Trial: The number of heads in 100 tosses of a coin.

$Pr[\text{head}] = 0.75$. Lines: The real probability (in red); Markov bound (in blue); Chernoff bound (in green).
LP+Random rounding paradigm: MaxSAT problem
A randomized approximation algorithm for Max3Sat

**INPUT:**
Given a set of clauses $C_1, \ldots, C_k$, each of length 3, over a set of boolean variables $X = \{x_1, \ldots, x_n\}$;

**OUTPUT:**
to find an assignment to maximize the number of satisfied clauses;

e.g.
$C_1 : x_1 \lor \neg x_2 \lor x_3$
$C_2 : \neg x_1 \lor \neg x_2 \lor x_4$
$C_3 : \neg x_3 \lor \neg x_4 \lor x_5$
$C_4 : \neg x_2 \lor \neg x_4 \lor x_7$
Algo1 (remarkably simple):

1. set each variable $x_i$ to 1 with probability $\frac{1}{2}$. 
The expected number of clauses satisfied by Algo1 is within an approximation factor $\frac{7}{8}$ of optimal.

Proof.

- Let $X$ be the number of satisfied clauses. $X_i$ is an index variable such that $X_i = 1$ if $C_i$ was satisfied, and $X_i = 0$ otherwise.

- $X = X_1 + \ldots + X_k$

- $E(X) = E(X_1 + \ldots + X_k) = E(X_1) + \ldots + E(X_k) = \frac{7}{8}k$

- $E(X_i) = \frac{7}{8}$

- and we have a lower bound: $OPT \leq k$.

- Thus, $E(X) \geq \frac{7}{8}OPT$. 

\[ \square \]
Probability method:

**Corollary**

*There exists at least an assignment to satisfy at least $\frac{7}{8}k$ clauses.*

(Intuition: the expectation is over $\frac{7}{8}k$ clauses. Just an existence proof.)
Probability method again:

**Corollary**

*All 3SAT instance with at most 7 clauses are satisfied.*

(Intuition: The unsatisfied clause number is \( \frac{1}{8} k = \frac{7}{8} < 1 \).)
Question: how to find an assignment to satisfy at least $\frac{7}{8}k$ clauses?

Algo 2:

1. repeat Algo1 until at least $\frac{7}{8}k$ clauses are satisfied.
The expected running time of Algo2 is polynomial. In particular, the expected number of repetition is less than $8k$. 
Proof.

- Let $p$ be the probability that at least $\frac{7}{8}$ clauses are satisfied;
- It suffices to prove that $\frac{1}{p} \leq 8k$. (Reason: the expected waiting time of an event with probability $p$ is $\frac{1}{p}$.)
- Let $p_j$ be the probability that EXACTLY $j$ clauses are satisfied. We have:
  1. $p = \sum_{j \geq \frac{7}{8}k} p_j$,
  2. $p + \sum_{j < \frac{7}{8}k} p_j = 1$;
  3. $E(X) = \frac{7}{8}k = \sum_{j=1}^{k}jp_j = \sum_{j \geq \frac{7}{8}k} jp_j + \sum_{j < \frac{7}{8}k} jp_j$.

Thus,

$$\frac{7}{8}k \leq \sum_{j \geq \frac{7}{8}k} kp_j + \sum_{j < \frac{7}{8}k} k'p_j = kp + k'(1 - p) \leq k' + kp.$$ 

($k'$ is the max number such that $k' < \frac{7}{8}k$.)

Thus, $kp \geq \frac{7}{8}k - k'$, and $p \geq \frac{1}{8k}$. (since $\frac{7}{8}k - k' \geq \frac{1}{8}$.)
Algo3: “LP+Random Rounding” strategy
ILP formulation

\[ C1 : x_1 \lor \neg x_2 \lor x_3 \]
\[ C2 : \neg x_1 \lor \neg x_2 \lor x_4 \]
\[ C3 : \neg x_3 \lor \neg x_4 \lor x_5 \]
\[ C4 : \neg x_2 \lor \neg x_4 \lor x_7 \]

ILP:

\[
\begin{align*}
\text{max} & \quad z = z_1 + z_2 + \ldots z_k \\
\text{s.t.} & \quad x_1 + (1 - x_2) + x_3 \geq z_1 \\
& \quad (1 - x_1) + (1 - x_2) + x_4 \geq z_2 \\
& \quad (1 - x_3) + (1 - x_4) + x_5 \geq z_3 \\
& \quad x_i = 0/1 \\
& \quad z_j = 0/1
\end{align*}
\]
LP:

\[
\begin{align*}
\text{max} \quad z &= z_1 + z_2 + \ldots + z_k \\
\text{s.t.} \quad &x_1 + (1 - x_2) + x_3 \geq z_1 \\
& (1 - x_1) + (1 - x_2) + x_4 \geq z_2 \\
& (1 - x_3) + (1 - x_4) + x_5 \geq z_3 \\
& \ldots \\
& x_i \leq 1 \\
& z_j \leq 1
\end{align*}
\]
Algo3:

1. Let $x^*$, $z^*$ denote the optimal solution to LP.
2. Randomly set variable $x_i = TRUE$ with probability $x_i^*$. 
Theorem

A clause $C_j$ is satisfied with a probability at least $(1 - (1 - \frac{1}{3})^3)z^*_j$.

Proof.

Suppose w.l.o.g $C_j = x_1 \lor x_2 \lor x_3$. We have:

\[
Pr(C_j \text{ is satisfied}) = 1 - (1 - x_1^*)(1 - x_2^*)(1 - x_3^*) \\
\geq 1 - (\frac{1}{3}((1 - x_1^*) + (1 - x_2^*) + (1 - x_3^*))^3 \\
\geq 1 - (1 - \frac{1}{3}z^*_j)^3 \quad (\text{by Fact 2.}) \\
\geq (1 - (1 - \frac{1}{3})^3)z^*_j \quad (\text{by Fact 3.})
\]
Some facts

- **Fact 1:** The optimal solution satisfies: \( x_1^* + x_2^* + x_3^* \geq z_j^* \).
- **Fact 2:** \( (x_1x_2...x_n)^{\frac{1}{n}} \leq \frac{1}{n}(x_1 + x_2 + ... + x_n) \)
- **Fact 3:** \( f(x) = 1 - \left(1 - \frac{1}{3}x\right)^3 \) is concave, and greater than \( g(x) = (1 - (1 - \frac{1}{3})^3)x \) at the two ends of \([0, 1]\). Thus, \( f(x) \geq g(x) \) for any \( x \in [0, 1] \).
Theorem

(Goemans, W ’94) Algo3 is a \((1 - \frac{1}{e})\)-approximation algorithm, where \((1 - \frac{1}{e}) = 0.632\).

Proof.

Let \(X\) be the number of satisfied clauses. Let index variable \(c_j\) be 1 when clause \(C_j\) is satisfied, and 0 otherwise. Thus, \(X = c_1 + c_2 + \ldots + c_k\).

\[
E(X) = E(c_1) + E(c_2) + \ldots + E(c_k)
\]
\[
= \sum_j Pr(C_j \text{ is satisfied}) \quad \text{(previous theorem)}
\]
\[
\geq \sum_j (1 - (1 - \frac{1}{3})^3)z_j^* 
\]
\[
\geq (1 - (1 - \frac{1}{3})^3) \sum_j z_j^* 
\]
\[
= (1 - (1 - \frac{1}{3})^3)z_{LP} 
\]
\[
\geq (1 - (1 - \frac{1}{3})^3)OPT \quad \text{(by } z_{LP} \geq z_{ILP} = OPT) 
\]
\[
\geq (1 - \frac{1}{e})OPT \quad \text{(by } (1 - \frac{1}{n})^n \leq \frac{1}{e})
\]
LP + Random rounding paradigm: VLSI DESIGN problem
A *gate-array* is a two-dimensional $\sqrt{n} \times \sqrt{n}$ array of gates abutting each other.

A *net* is a set of gates to be connected by a wire. In our problem, the number of gates in a set is exactly 2.

Assume that the wire for each net contains at most one $90^\circ$ turn, called “one-bend” route. Thus, in joining the two end-points of a net, the wire will either first traverse the horizontal dimension and then the vertical dimension, or the other way around. In particular, a net, which connects two gates in the same column or the same row, only has one choice.

Let $w_S(b)$ denote the number of wires that pass through boundary $b$ in a solution $S$. Here, each of the four edges of a grid is called a boundary $b$. 
The Problem is $\min_S \max_b w_S(b)$. (Intuition: not too many wires pass through any boundary.)
This problem can be cast as a $0 - 1$ linear program (because for each net, there is at most 2 choices.).

For each net $i$ from left end-point to the right end-point, we define 2 variables $x_{i0}$ and $x_{i1}$ to describe the direction of the wire:

- $x_{i0} = 1, x_{i1} = 0$ if net $i$ goes horizontally first
- $x_{i0} = 0, x_{i1} = 1$ if net $i$ goes vertically first

For each boundary $b$ in the array, let

- $T_{b0} = \{i | \text{net } i \text{ passes through } b \text{ if } x_{i0} = 1\}$
- $T_{b1} = \{i | \text{net } i \text{ passes through } b \text{ if } x_{i1} = 1\}$
With these definitions, our integer program can be expressed as:

\[
\begin{align*}
\min \ w \\
\sum_{i \in T_{b_0}} x_{i0} + \sum_{i \in T_{b_1}} x_{i1} &= 1 \quad \forall i \\
\sum_{i \in T_{b_0}} x_{i0} + \sum_{i \in T_{b_1}} x_{i1} &\leq w \quad \forall b \\
x_{i0}, x_{i1} &\in \{0, 1\}
\end{align*}
\]

Let \( OPT \) be the objective value of the above \( ILP \).
We solve instead the linear program relaxation of \( ILP \) by replacing \( x_{i0}, x_{i1} \in \{0, 1\} \) to \( x_{i0}, x_{i1} \in [0, 1] \):

\[
\begin{align*}
\min w \\
x_{i0} + x_{i1} & = 1 \quad \forall i \\
\sum_{i \in T_{b0}} x_{i0} + \sum_{i \in T_{b1}} x_{i1} & \leq w \quad \forall b \\
x_{i0}, x_{i1} & \in [0, 1]
\end{align*}
\]

Let \( \hat{x}_{i0} \) and \( \hat{x}_{i1} \) be the solution, \( \hat{w} \) be the objective value, of the above \( LP \). Obviously, \( \hat{w} \leq OPT \).
Algo: Randomized Rounding $\hat{x}_{i0}$ and $\hat{x}_{i1}$ to 0 and 1.

Indepently for each $i$, define 2 random variables, $\bar{x}_{i0}$ and $\bar{x}_{i1}$.

$$Pr(\bar{x}_{i0} = 1, \bar{x}_{i1} = 0) = \hat{x}_{i0}$$
$$Pr(\bar{x}_{i1} = 1, \bar{x}_{i0} = 0) = \hat{x}_{i1}$$

Obviously, $E(\bar{x}_{i0}) = \hat{x}_{i0}$ and $E(\bar{x}_{i1}) = \hat{x}_{i1}$.

Now we get a solution $S = \{\hat{x}_{i0}, \hat{x}_{i1}, i = 1, 2, ..., n\}$ to the problem, how about its performance?
Theorem

Let $\epsilon$ be a real number such that $0 < \epsilon < 1$. Then with probability $1 - \epsilon$, the solution $S$ produced by randomized rounding satisfies:

$$w_S \leq (1 + \Delta(\hat{w}, \epsilon/2n))\hat{w} \leq (1 + \Delta(OPT, \epsilon/2n))OPT.$$

where $\Delta(\mu, \epsilon)$ is defined as: if let $\left[\frac{e^\delta}{(1+\delta)^{1+\delta}}\right]^\mu = \epsilon$, then $\delta = \Delta(\mu, \epsilon)$.

Proof

- The second inequality is obvious.
- In order to prove the first inequality, we just need to prove that: for any boundary $b$, the probability that $w_S(b) > \hat{w}(1 + \Delta(\hat{w}, \epsilon/2n))$ is at most $\epsilon/2n$. (Why?)

Consider a boundary $b$, $w_S(b) = \sum_{i \in T_{b_0}} x_{i0} + \sum_{i \in T_{b_1}} x_{i1}$ then

$$E(w_S(b)) = \sum_{i \in T_{b_0}} E(x_{i0}) + \sum_{i \in T_{b_1}} E(x_{i1}) = \sum_{i \in T_{b_0}} \hat{x}_{i0} + \sum_{i \in T_{b_1}} \hat{x}_{i1} \leq \hat{w}$$
According to the Definition of $\Delta$ and Chenoff Bound, we have \[ Pr(w_S(b) > \hat{w}(1 + \Delta(\hat{w}, \epsilon/2n))) \leq \epsilon/2n \] and the theorem follows.
Randomized divide-and-conquerer
Randomized divide-and-conquerer: Selection problem

**INPUT:**
Given a set of number $S = \{a_1, a_2, \ldots, a_n\}$, and a number $k \leq n$;

**OUTPUT:**
the median in $S$, or the $k$-th smallest item.

Note: known deterministic linear algorithms, say Blum ’73 ($16n$ comparisons), and D. Zuick ’95 ($2.95n$ comparisons).
Randomized algorithm:
Randomized divide-and-conquerer: \textsc{Selection} problem

\begin{verbatim}
Select( n, k)

Choose an element \( a_i \) in \( S \) uniformly at random;

for j = 1 to n
   do
      if \( a_j > a_i \)
         add \( a_j \) onto \( S^+ \);
      else
         add \( a_j \) onto \( S^- \);
   done

if |\( S^- \)| = k-1
   return \( a_i \);
else
   if |\( S^- \)| \( \geq \) k
      Select( \( S^- \), k);
   else
      Select(\( S^+ \), k-1-l);
   //Here, l = |\( S^- \)|
\end{verbatim}
Randomized divide-and-conquer: \textbf{Selection} problem

(\textbf{Intuition:} sloving an extension of the original problem, i.e., to find the $k$-th median. At first, an element $a_i$ is chosen to split $S$ into two parts $S^+ = \{a_j : a_j \geq a_i\}$, and $S^- = \{a_j : a_j < a_i\}$. We can determine whether the $k$-th median is in $S^+$ or $S^-$. Thus, we perform iteration on ONLY one subset.)

\textbf{Difficulty:} how to choose the splitter?

- Bad choice: select the smallest element at each iteration.

\[ T(n) = T(n - 1) + O(n) = O(n^2) \]
Randomized divide-and-conquer: Selection problem

- Ideal choice: select the median at each iteration.
  \[ T(n) = T\left(\frac{n}{2}\right) + O(n) = O(n) \]

- Good choice: select a “centered” element \( a_i \), i.e., \( |S^+| \geq \epsilon n \), and \( |S^-| \geq \epsilon n \) for a fixed \( \epsilon > 0 \).
  \[ T(n) \leq T((1 - \epsilon)n) + O(n) \leq \epsilon n + c(1 - \epsilon)n + c(1 - \epsilon)^2n + \ldots = O(n). \]

E.g.: \( \epsilon = \frac{1}{4} \):

```
\[
\begin{array}{cccccccccccc}
S1 & S2 & S3 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & Sn \\
1/4n & & & & & & & & & & \\
\downarrow \hspace{1cm} & \downarrow \hspace{1cm} & \downarrow \hspace{1cm} & \downarrow \hspace{1cm} & \downarrow \hspace{1cm} & \downarrow \hspace{1cm} & \downarrow \hspace{1cm} & \downarrow \hspace{1cm} & \downarrow \hspace{1cm} & \downarrow \hspace{1cm} & \downarrow \hspace{1cm} \\
1/2n & & & & & & & & & & \\
\end{array}
\]
```
Key observation: if we choose a splitter \( a_i \in S \) uniformly at
random, it is easy to get a good splitter since a fairly large fraction
of the elements are “centered”.

**Theorem**

*The expected running time of Select\((n,k)\) is \(O(n)\).*
Proof.

Let $\epsilon = \frac{1}{4}$. We’ll say that the algorithm is in phase $j$ when the size of set under consideration is in $[n(\frac{3}{4})^{-j}, n(\frac{3}{4})^j]$.

Let $X$ be the number of steps. And $X_j$ be the number of steps in phase $j$. Thus, $X = X_0 + X_1 + \ldots$.

Consider the $j$-th phase. The probability to find a centered splitter is $\geq \frac{1}{2}$ since at least half elements are centered. Thus, the expected number of iterations to find a centered splitter is: $2$.

Each iteration costs $cn(\frac{3}{4})^j$ steps since there are at most $n(\frac{3}{4})^j$ elements in phase $j$. Thus, $E(X_j) \leq 2cn(\frac{3}{4})^j$.

$E(X) = E(X_0 + X_1 + \ldots) \leq \sum_j 2cn(\frac{3}{4})^j \leq 8cn$. 

\[\square\]
(See extra slides)